



Enhancing Dynamic CT Image Reconstruction with Neural Fields and Optical Flow

Maths4DL Conference on Inverse Problems and Deep Learning

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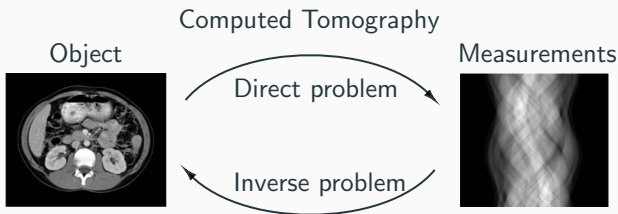
University of Bath

Introduction

Static Inverse Problems

- Given the consequences/measurements f , to determine the causes/object $u : \Omega \subset \mathbb{R}^d \rightarrow \mathbb{R}$.

$$\underbrace{K}_{\text{Forward model}} \underbrace{u}_{\text{Object}} + \underbrace{\varepsilon}_{\text{Noise}} = \underbrace{f}_{\text{Measurements}}$$



$$Ku = \int_L u(x) dx$$

X-ray Transform

Dynamic Inverse Problems

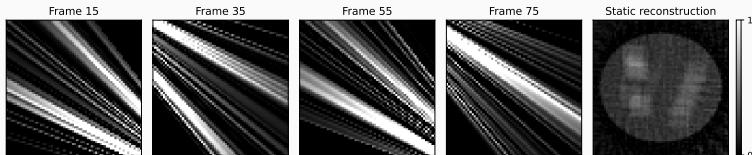
- We have measurements $\{f_{t_1}, \dots, f_{t_{N_T}}\} \subset \mathbb{R}^M$ collected at times $0 = t_1 < \dots < t_{N_T} = T$ from M sensors. We seek $u : \Omega \times [0, T] \rightarrow \mathbb{R}$.

$$K_t[u_t] + \varepsilon_t = f_t, \quad \text{for } t \in \{t_1, \dots, t_{N_T}\}.$$

- Each f_t is *highly undersampled in space!* (no gating or binning).



Time



Discretisation and Regularity

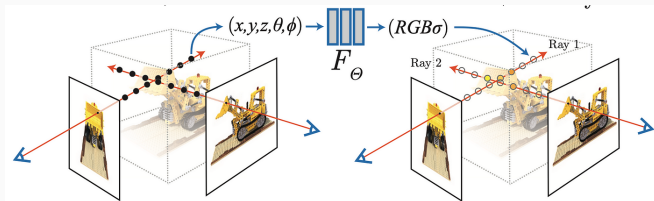
- $K_t[u_t] + \varepsilon_t = f_t \rightarrow \mathbf{K}_t[U_t] + \varepsilon_t = f_t$, where $U \in \mathbb{R}^{N \times N_T}$.
- Time coherence? Variational regularisation, e.g., $\|U\|_*$, $\|WU\|_1$.
- *PDE-based motion models* are an alternative. It is an equation of the form $r(u, v) = 0$, where v is the velocity field.
- *Optical flow equation*: Brightness constancy assumption¹.
- $u(x(t), t) \sim u(x(t + \delta), t + \delta) \Rightarrow \partial_t u + v \cdot \nabla u = 0$.

¹B. Horn, B. Schunk, Artificial Intelligence, 1981

Neural Fields

Neural fields

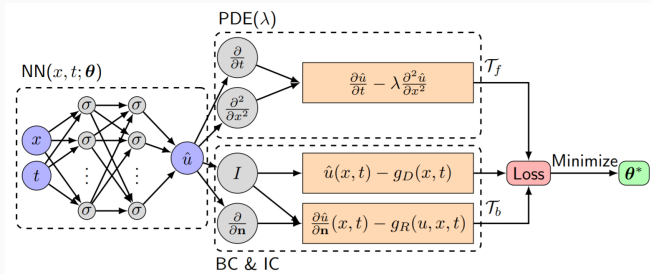
Neural Fields² have become popular for Computer Vision tasks and solving Partial Differential Equations (PINNs).



²B. Mildenhall et al., 2021.

Neural fields

Neural Fields² have become popular for Computer Vision tasks and solving Partial Differential Equations (PINNs³).

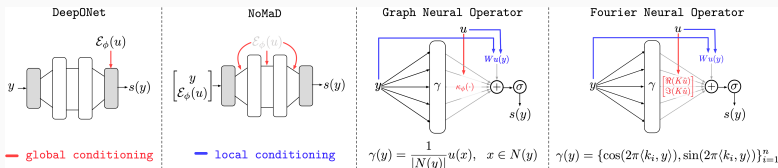


²B. Mildenhall et al., 2021.

³M. Raissi, P. Perdikaris, G. Karniadakis, 2019

Parenthesis on Operator Learning

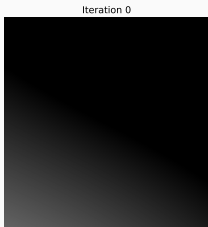
- Neural fields and PINNs are instance-based. Do not fit in the learning framework!
- Conditioned Neural Fields and Operator Learning offer an option to learn from data⁴.



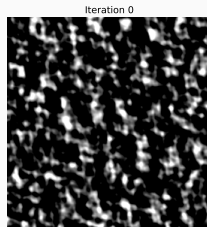
⁴S. Wang et al, CViT: Continuous Vision Transformer for Operator Learning, 2024

Neural Fields

- Complexity and memory depend on the weights → **Neural fields overcome the curse of dimensionality!**⁵
- Offer a lightweight, continuous, and differentiable representation.
- Spectral bias: MLPs introduce implicit regularisation (SIREN or Fourier features mapping to overcome this).



No Fourier feature embedding

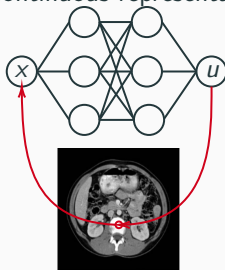
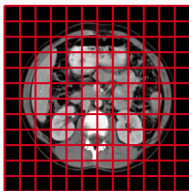


Fourier feature embedding

⁵M. Hutzenthaler, A. Jentzen, T. Kruse, T. Anh, SN Partial Differential Equations and Applications, 2020.

Neural Fields

Grid-based representation Continuous representation



	Mesh free	Curse of Dimensionality	Regularity	Convergence guarantees
Grid-based	✗	✗	✗	✓
Neural Field	✓	✓	✓	✗

- How do neural fields compare to grid-based representations in the context of dynamic inverse problems?
- What is the effect of motion models for neural fields?

Variational Problem

Variational Problem

- $K_t[u_t] + \varepsilon_t = f_t$, for $t \in \{t_1, \dots, t_{N_T}\}$.
- $\mathcal{D}(u, f) := \sum_{i=1}^{N_T} \frac{1}{2} \|K_{t_i}[u_{t_i}] - f_{t_i}\|_2^2$.
- Optical flow equation: $\partial_t u + v \cdot \nabla u = 0$, in $\Omega_T := \Omega \times [0, T]$.

Joint Image Reconstruction and Motion Estimation

$$\min_{u, v} \mathcal{D}(u, f) + \alpha \mathcal{R}_1(u) + \beta \mathcal{R}_2(v) + \gamma \int_{\Omega_T} |\partial_t u + v \cdot \nabla u|.$$

- $\mathcal{R}_1(u) := \int_0^T \text{TV}(u_t)$, $\mathcal{R}_2(v) := \int_0^T \text{TV}(v_t)$

Variational problem

The variational problem

Represent image and velocity with two neural fields:

$$\min_{\theta, \phi} \mathcal{D}(u_{\theta}, f) + \alpha \mathcal{R}_1(u_{\theta}) + \beta \mathcal{R}_2(v_{\phi}) + \gamma \int_{\Omega_T} |\partial_t u_{\theta} + v_{\phi} \cdot \nabla u_{\theta}|.$$

Current methods with neural fields

Literature relies on the regularity of neural fields:

$$\min_{\theta} \mathcal{D}(u_{\theta}, f) + \alpha \mathcal{R}_1(u_{\theta}) + \beta \mathcal{R}_2(v_{\phi}) + \gamma \int_{\Omega_T} |\partial_t u_{\theta} + v_{\phi} \cdot \nabla u_{\theta}|.$$

Dynamic MRI: J. Kunz, S. Ruschke, R. Heckel, 2023; T. Catalán et al., 2022.

Dynamic CT: A. Reed et al. 2021.

Dynamic Photoacoustic Tomography: L. Lozenski et al., 2024.

Variational problem

The variational problem

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Proposed method

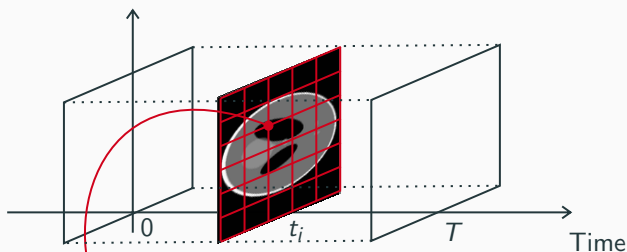
We incorporate explicit motion regularisers:

$$\min_{\theta, \phi} \mathcal{D}(u_{\theta}, f) + \alpha \mathcal{R}_1(u_{\theta}) + \beta \mathcal{R}_2(v_{\phi}) + \gamma \int_{\Omega_T} |\partial_t u_{\theta} + v_{\phi} \cdot \nabla u_{\theta}|.$$

Numerical evaluation - Neural Field

- Closed form for $K_t[(u_\theta)_t] = \int_{L(t)} (u_\theta)_t$? **X**

- $\mathcal{D}(u_\theta, f) = \frac{1}{N_T} \sum_{i=1}^{N_T} \frac{1}{2} \|\mathbf{K}_{t_i}[(U_\theta)_{t_i}] - f_{t_i}\|_2^2$



$$(U_\theta)_{t_i} := \{u_\theta(\mathbf{x}_l, t_i)\}_l \rightarrow \mathbf{K}_{t_i}[(U_\theta)_{t_i}]$$

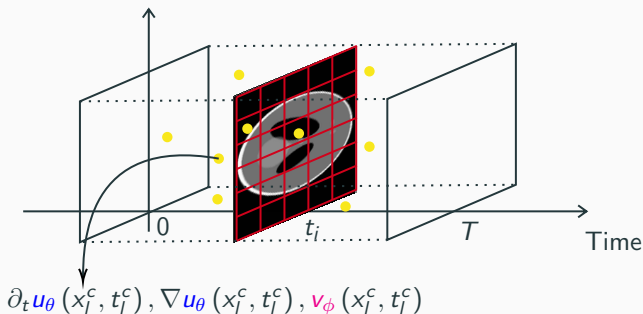
$\mathbf{x}_l$: Points on a cartesian grid

- Memory efficiency: we can proceed by mini-batches in time!

Numerical evaluation - Neural Field

- Motion regulariser term is estimated via Monte Carlo as in PINNs.
- We randomly sample collocation points $\{(x_l^c, t_l^c)\}_l \subset \Omega_T$.

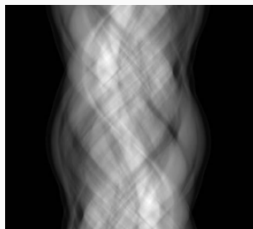
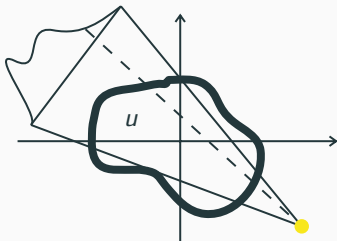
$$\int_{\Omega_T} |\partial_t \mathbf{u}_\theta + \mathbf{v}_\phi \cdot \nabla \mathbf{u}_\theta| \approx \frac{|\Omega_T|}{N_c} \sum_l |\partial_t \mathbf{u}_\theta(x_l^c, t_l^c) + \mathbf{v}_\phi(x_l^c, t_l^c) \cdot \nabla \mathbf{u}_\theta(x_l^c, t_l^c)|$$



Numerical Experiments: Regularised against unregularised neural fields

Dynamic Computed Tomography

- Fan-beam projections:



Sinogram

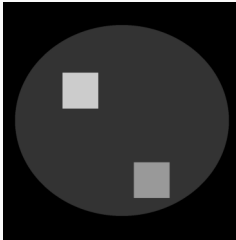
- Ground-truth phantom: 100 frames, 1024^2 spatial resolution.
- Dynamic setting: one random projection per time instant⁶.
- Reconstruction: 100 frames, 64^2 spatial resolution.
- Tomosipo: ASTRA-toolbox⁷ + PyTorch.

⁶M. Burger, H. Dirks, L. Frerking, A. Hauptmann, T. Helin, S. Siltanen, 2017

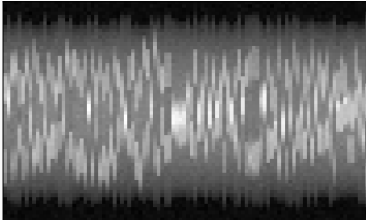
⁷W. Aarle et al., 2015 & 2016; A. Hendriksen et al., 2021

The phantom

The phantom

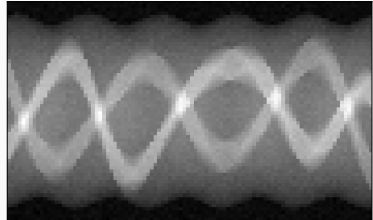


Random sampling



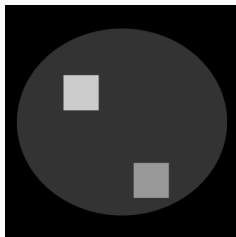
Time

Sequential sampling

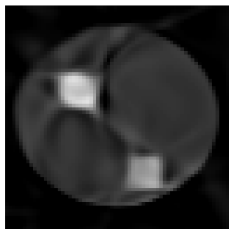


Time

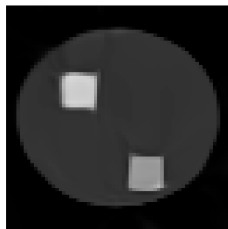
Results - Unregularised vs Regularised neural fields



Ground Truth



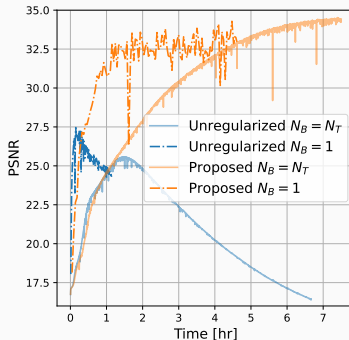
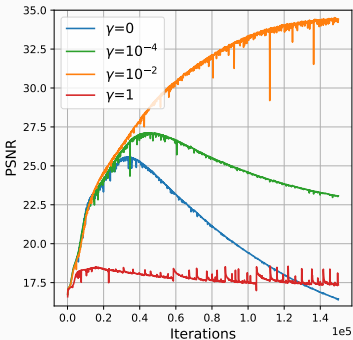
Unregularised solution
 $(\alpha, \beta, \gamma) = (0, 0, 0)$



Regularised solution
 $(\alpha, \beta, \gamma) = (0, 0, 10^{-2})$

Results - Unregularised vs Regularised neural fields

- Explicit motion regularisers achieve higher PSNR and stabilise training.
- Mini-batch accelerates optimisation at the cost of higher variability.



Numerical Experiments:
Regularised neural fields against
grid-based methods

Grid-based approach

The variational problem

$$\min_{u,v} \mathcal{D}(u, f) + \alpha \mathcal{R}_1(u) + \beta \mathcal{R}_2(v) + \gamma \|\partial_t u + v \cdot \nabla u\|_1.$$

We use finite differences and an alternating scheme + PDHG. Let (u^k, v^k) be the k -th iteration.

Update u^{k+1} : fix v^k and solve

Problem in u

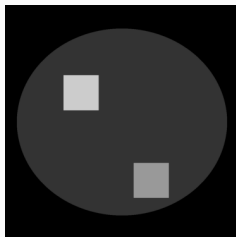
$$\min_u \mathcal{D}(u, f) + \alpha \mathcal{R}_1(u) + \gamma \|\partial_t u + v^k \cdot \nabla u\|_1.$$

Update v^{k+1} : fix u^{k+1} and solve

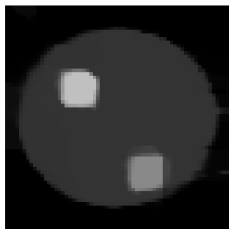
Problem in v

$$\min_v \beta \mathcal{R}_2(v) + \gamma \|\partial_t u^{k+1} + v \cdot \nabla u^{k+1}\|_1.$$

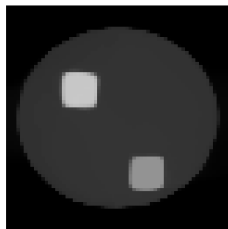
Results - Grid-based vs Neural fields for random sampling



Ground Truth



Grid-based solution, PSNR: 27.09
 $(\alpha, \beta, \gamma) = (10^{-3}, 10^{-4}, 10^{-3})$

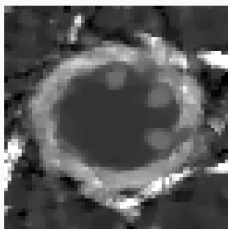


Neural field solution, PSNR: **32.92**
 $(\alpha, \beta, \gamma) = (10^{-3}, 10^{-4}, 10^{-3})$

Results - Grid-based vs Neural fields for random sampling



Ground Truth

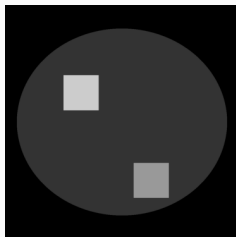


Grid-based solution, PSNR: 17.55
 $(\alpha, \beta, \gamma) = (10^{-4}, 10^{-4}, 5 \times 10^{-3})$

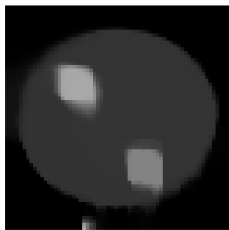


Neural field solution, PSNR: **28.88**
 $(\alpha, \beta, \gamma) = (0, 0, 10^{-2})$

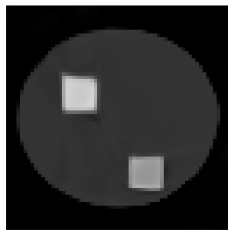
Results - Grid-based vs Neural fields for sequential sampling



Ground Truth



Grid-based solution, PSNR: 22.65
 $(\alpha, \beta, \gamma) = (10^{-3}, 10^{-4}, 10^{-3})$



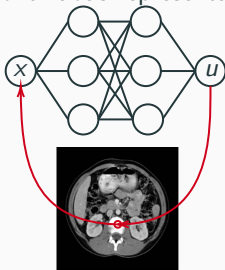
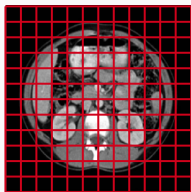
Neural field solution, PSNR: **26.42**
 $(\alpha, \beta, \gamma) = (0, 0, 10^{-2})$

Conclusions

Conclusions

- Neural Fields for highly undersampled dynamic CT.
- Neural fields can outperform grid-based representations.
- Explicit motion regularisers in neural fields enhance reconstruction quality. Additionally, motion is estimated.
- Neural fields overcome the curse of dimensionality and are implicitly regularised $\rightarrow$ large scale problems in imaging.
- Drawback: Instance-based method. Computational cost.

Grid-based representation Continuous representation



Comments on memory consumption

- In terms of memory, neural fields are suitable for large scale dynamic imaging inverse problems.

		Spatial resolution				
		64 × 64	128 × 128	256 × 256	512 × 512	1024 × 1024
3 hidden layers 128 neurons each	Grid-based	0.073	0.264	1.039	4.109	-
	Neural Field Sampling rate: 0	0.030	0.063	0.204	0.77	3.02
	Neural Field Sampling rate: 0.1	0.033	0.073	0.245	0.928	3.66

Memory consumption in GiB.

- Data-fidelity term: one frame sampled per iteration
- Sampling rate: $\frac{\# \text{ Collocation points}}{\# \text{ Spatial pixels}}$.